# ON THE EXISTENCE AND CLASSIFICATION OF DIFFERENTIABLE EMBEDDINGS

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## §1. INTRODUCTION

LET $M$ be a compact $k$-connected differential $n$-manifold without boundary. Our object is to prove, under suitable restrictions on $k$ and $n$, an existence theorem for embedding $M$ in the Euclidean space $R^{2n-k-1}$ (Theorem (2.3)), and a classification theorem for isotopy classes of embeddings of $M$ in $R^{2n-k}$ if $M$ is orientable (Theorem (2.4)). This is done by first proving Theorems (2.1) and (2.2) which reduce the embedding problems to questions involving immersions, and then applying the theory of immersions [2].

A particular case of (2.3) is the following:

THEOREM (1.1). *If $n > 4$, $M$ is embeddable in $R^{2n-1}$ if and only if its normal Stiefel–Whitney class $\overline{W}^{n-1}$ vanishes.*

Massey [5, 6, 7] has shown that if $\overline{W}^{n-1} \neq 0$, then $M$ is non-orientable and $n$ is a power of 2. Thus we obtain:

THEOREM (1.2). *If $n > 4$ and $M$ is orientable, $M$ is embeddable in $R^{2n-1}$.*

This is also true if $n = 3$; see [4]. The case $n = 4$ is unsolved, even if $M$ is simply connected. However, Smale has proved (unpublished) that every homotopy 4-sphere is embeddable in $R^5$.

It should be remarked that the existence Theorems (2.1) and (2.3) apply to both orientable and non-orientable manifolds, but the classification Theorems (2.2) and (2.4) apply only to orientable manifolds.

(1.3). DEFINITIONS AND NOTATION. All manifolds considered here are differential. The boundary of a manifold $X$ is $\partial X$. We put $X - \partial X = \text{int } X$.

An *immersion* of an $n$-manifold $X$ in Euclidean $v$-space $R^v$ is a differentiable map $f: X \to R^v$ of rank $n$ everywhere. An *embedding* is an immersion which is 1–1. If $f$ and $g$ are immersions of $X$ in $R^v$, a *regular homotopy connecting* $f$ to $g$ is a differentiable homotopy $F: X \times I \to R^v$ such that $F_0 = f$, $F_1 = g$, and each $F_t$ is an immersion. If in addition each $F_t$ is an embedding, then $F$ is an *isotopy*.

If $F, G : X \times I \to R^v$ are regular homotopies, we say that $F$ and $G$ are *regularly homotopic* if there is a differentiable map $H : X \times I \times I \to R^v$ such that for each $t \in I$ the map $H_t$ is a regular homotopy, where $H_t(x, s) = H(x, s, t)$, and if $H_0 = F$, $H_1 = G$.

An *immersion of $X$ in $R^v$ with a normal vector field* is a pair $(g, \mu)$ where $g : X \to R^v$ is an immersion, and $\mu : X \to R^v$ is a differentiable map such that for each $x \in X$, $\mu(x)$ is a unit vector orthogonal to the image (under the differential of $g$) of the tangent plane to $X$ at $x$. Two such pairs $(f, v)$ and $(g, \mu)$ are regularly homotopic if there is a regular homotopy $h_t$ connecting $f$ to $g$, and a homotopy $\lambda_t : M \to R^v$ connecting $v$ to $\mu$, such that for each $t$, $(f_t, \lambda_t)$ is an immersion with normal vector field.

If a cycle $u$ bounds, we write $u \sim 0$.

Homology and cohomology groups have integer coefficients unless other coefficients are indicated.

If $X$ is a manifold, the normal Stiefel–Whitney classes of $X$ are denoted by $\overline{W}^i$. These are $i$-dimensional cohomology classes with coefficients as follows: $Z_2$ if $i$ is even, $Z$ if $i$ is odd and $X$ is orientable, twisted integers if $i$ is odd and $X$ is non-orientable.

## §2. THE MAIN RESULTS

Let $M$ be a compact $k$-connected differential manifold without boundary. Let $M_0$ denote $M$ minus a point.

THEOREM (2.1). *Assume $0 \le k < \frac{1}{2}(n-4)$. If $M_0$ can be immersed in $R^{2n-k-1}$ with a normal vector field, then $M$ can be embedded in $R^{2n-k-1}$.*

It is easy to prove the converse if $M$ is orientable, without any restriction on $k$, using (2.3) below.

THEOREM (2.2). *Assume $0 \le k \le \frac{1}{2}(n-4)$. If $M$ is orientable there is a 1–1 correspondence between the isotopy classes of embeddings of $M$ in $R^{2n-k}$ and the regular homotopy classes of immersions of $M_0$ in $R^{2n-k}$ with a normal vector field.*

The proofs of Theorems (2.1) and (2.2) are postponed until §4.

Let $T_{m,n+1}$ be the bundle associated to the frame bundle of $M_0$ with fibre the Stiefel manifold $V_{m,n+1}$ of $(n+1)$-frames in $R^m$, the linear group in $n$ variables acting in the natural way on the first $n$ vectors of a frame. According to [2], the existence of an immersion of $M_0$ in $R^m$ with a normal vector field is equivalent to the existence of a section of $T_{m,n+1}$. Moreover, it is easy to prove, using [2], that the regular homotopy classes of immersions of $M_0$ in $R^m$ with a normal vector field are in 1–1 correspondence with the homotopy classes of sections of $T_{m,n+1}$.

If $M$ is $k$-connected, the only obstruction to constructing a section of $T_{2n-k-1,n+1}$ is the normal Stiefel–Whitney class $\overline{W}^{n-k-1}$ of $M_0$ (or $M$). If $M$ is orientable, the homotopy classes of sections of $T_{2n-k,n+1}$ are in 1–1 correspondence with the elements of $H^{n-k-1}(M, \pi_{n-k-1}(V_{2n-k,n+1}))$. Therefore we obtain the following corollaries of (2.1) and (2.2).†

† J. P. Levine has proved a similar theorem in the orientable case (*Not. Amer. Math. Soc.* 9 (1962), 220).

THEOREM (2.3). *If $0 \leq k < \frac{1}{2}(n-4)$, a compact unbounded k-connected n-manifold M can be embedded in $R^{2n-k-1}$ if and only if its normal Stiefel–Whitney class $\overline{W}^{n-k-1}$ vanishes.*

THEOREM (2.4). *If $0 \leq k \leq \frac{1}{2}(n-4)$, the isotopy classes of embeddings of an orientable compact unbounded k-connected manifold M in $R^{2n-k}$ are in 1–1 correspondence with the elements of* $\begin{cases} H_{k+1}(M; Z) & \text{if } n-k \text{ is odd;} \\ H_{k+1}(M; Z_2) & \text{if } n-k \text{ is even.} \end{cases}$

## §3. MATERIAL USED

In the proofs of (2.1) and (2.2) we shall use the following two embedding theorems. Recall that $M_0 = M$ minus a point.

THEOREM (3.1). *Let M be a k-connected n-manifold*

(a) *If $v \geq 2n-k-1$, then $M_0$ can be immersed in $R^v$, and any immersion is regularly homotopic to an embedding.*

(b) *If $v \geq 2n-k$, any two embeddings f and g of $M_0$ in $R^v$ are regularly homotopic. If G is a regular homotopy connecting f and g, there is a regular homotopy $G_t$ of G such that $G_0 = G$, $G_1$ is an isotopy, and for each t, $G_t$ connects f to g.*

*Proof.* Part (a) is implicit in [3], and (b) can be proved by using the methods of [3]. The idea of the proof is that $M_0$ is diffeomorphic to a small neighborhood of an $(n-k-1)$-complex in $M$. Smale's theory of handles [8] can be used instead.

THEOREM (3.2). *Let X be a v-manifold and E an open n-disk.*

(a) *Suppose $2v \geq 3(n+1)$ and X is $(2n-v+1)$-connected. Let $g: E \to X$ be a proper map whose restriction to the complement of some compact set is an embedding. Then there is a homotopy, fixed outside of a compact set, which deforms g into an embedding.*

(b) *Suppose $2v > 3(n+1)$ and X is $(2n-v+2)$-connected. Let $g_0$ and $g_1: E \to X$ be proper embeddings which are connected by a homotopy fixed outside of a compact set. Then $g_0$ and $g_1$ are also connected by an isotopy $g_t$ fixed outside of a compact set.*

*Proof.* The proof is similar to the proofs of (4.1) and (5.1) of [1]. The only modification needed is to change remark (4.13) of [1] by replacing $\partial V$ with the complement of a suitable compact disk in $E$.

Let $B$ be the total space of a disk bundle over a manifold $N$ and let $A = \partial B$, so that $A$ is fibered by spheres. Identify $N$ with the zero section of $B$. The following facts are well known; cf. Thom [9], Whitney [10].

THEOREM (3.3).

(a) *The first obstruction to constructing a section of A is the cohomology class of N dual to the self-intersection of N in B.*

(b) *The corresponding interpretation for the obstruction $d(\sigma_0, \sigma_1)$ to deforming a section $\sigma_0$ of A into a section $\sigma_1$ of A is the cohomology class of N dual to the intersection in B of N with a homotopy of sections in B connecting $\sigma_0$ and $\sigma_1$.*

## §4. PROOFS OF (2.1) and (2.2)

(4.1). *Proof of* (2.1), *M orientable.* Let $f: M_0 \to R^{2n-k-1}$ be an immersion with a normal vector field $v$. By (3.1a), $f$ is regularly homotopic to an embedding; we can suppose therefore that $f$ is an embedding. Let $D_2 \subset M$ be an embedded closed disk of radius 2 with center $x_0$, and let $D_1$ be the concentric disk of radius 1. Let $E_2$ and $E_1$ be the interiors of $D_2$ and $D_1$. Put $M_1 = M - E_1$ and $M_2 = M - E_2$. We claim that $f(\partial M_1)$ is an $(n-1)$-sphere homotopic to zero in $X = R^{2n-k-1} - f(M_2)$. Let $\varepsilon$ be a positive number small enough to be the radius of a tubular neighborhood of $f(M_1)$. Let $\lambda : M_1 \to [0, \varepsilon]$ be a differentiable function equal to $\varepsilon$ on $M_2$ and to 0 on $\partial M_1$. Then $f(\partial M_1)$ bounds the image of $M_1$ by the map $x \to f(x) + \lambda(x)v(x)$, so that $f(\partial M_1) \sim 0$ in $X$. (We have used the orientability of $M$ to have $\partial M_1 \sim 0$ in $M_1$.)

Since $M$ is $k$-connected, Poincaré and Alexander duality shows that $H_i(X) = 0$ for $0 \leq i \leq n-2$, and a general position argument shows that $X$ is simply connected. Therefore the Hurewicz isomorphism between $\pi_{n-1}(X)$ and $H_{n-1}(X)$ shows that $f(\partial M_1)$ is homotopic to zero in $X$.

It is now possible to extend the map $f|M_1$ to a map $g: M \to R^{2n-k-1}$ such that $g(M_2) \cap g(E_2) = \varnothing$. Applying (3.2) to $g|E_2 : E_2 \to X$ leads to an embedding of $E_2$ in $X = R^{2n-k-1} - f(M_2)$ which agrees with $f$ outside of a compact neighbourhood of $\partial M_1$ in $E_2$. This embedding and $f|M_2$ thus fit together to form an embedding of $M$ in $R^{2n-k-1}$.

(4.2). *Proof of* (2.1), *M non-orientable.* Assume now that $k = 0$ and that $M$ is non-orientable. Keeping the notation of (4.1), we cannot conclude that $f(\partial M_1)$ is a boundary in $X$ but only that $f(\partial M_1)$ bounds mod 2. Equivalently, $f(\partial M_1)$ represents an even homology class in $X$.

We shall need an explicit cocycle $u_f$ representing the cohomology class $[u_f] \in H^{n-1}(M_2)$ that corresponds to the homology class of $f(\partial M_1)$ under Alexander duality. Such a cocycle is found in the following way. Let $C$ be an oriented singular disk in $R^{2n-1}$ bounded by $f(\partial M_1)$. For any $(n-1)$-simplex $\sigma$ in $M_2$ put $u_f(\sigma) = C \# f(\sigma) =$ intersection number of $C$ and $f(\sigma)$. As we observed above, $[u_f]$ is an even class; hence there are cochains $v$ and $w$ such that $u_f = 2v + \partial w$.

We shall prove that there is an embedding $g: M_1 \to R^{2n-1}$ such that $u_g = u_f - 2v$. It will follow that $[u_g] = 0$, and the rest of the proof proceeds as in (4.1).

We need the fact that $M_1$ can be described as a 'thickening' of an $(n-1)$-complex. This can be proved by using the techniques of [3], or Smale's theory of handles [8]. The interior of the singular disk $C$ will meet $f(M_1)$ only in the handles. It will then be a simple matter to change the embedding on one handle at a time, keeping track of the corresponding change in $u_f$. The point is that every time a handle pierces $C$, the boundary of $f(M_1)$ intersects $C$ twice.

For simplicity of notation, we assume that $M \subset R^{2n-1}$, and that $f$ is the inclusion map. Let $D^n$ be the closed unit $n$-ball. What we need from the theory of handles is that there exist a finite number of embeddings $h_i : D^{n-1} \times D^1 \to M_1$ with the following properties:

(1) $h_i(D^{n-1} \times \partial D^1) \subset \partial M_1$;

(2) $C \cap M_1 \subset \bigcup_i f_i((\text{int } D^{n-1}) \times D^1)$.

(The 'handles' are the sets $h_i(D^{n-1} \times D^1)$.) The cochain $u_f$ is now defined by the intersection numbers $C \# h_i(D^{n-1} \times 0)$.

Let us focus attention on a single handle $h_i(D^{n-1} \times D^1)$. We might as well assume that $h_i$ is the composite of the inclusion maps $D^{n-1} \times D^1 \subset D^{n-1} \times D^n \subset R^{2n-1}$, since we can bring this about by an isotopy of $R^{2n-1}$. A new embedding $g : M_0 \to R^{2n-1}$ is described as follows. Let $S^{n-1} = \partial D^n$, and let $P$ be the north pole of $S^{n-1}$, so that the handle $D^{n-1} \times D^1$ meets $D^{n-1} \times (\partial D^n)$ in $(D^{n-1} \times P) \cup (D^{n-1} \times (-P))$. Let $\alpha : (D^{n-1}, \partial D^{n-1}) \to (S^{n-1}, P)$ be a differentiable map, constant near $\partial D^{n-1}$. Define $g : M_1 \to R^{2n-1}$ by

$$g(x) = \begin{cases} x \text{ if } x \in D^{n-1} \times D^1 \\ (y, t\alpha(y)) \in D^{n-1} \times D^n \text{ if } x = (y, t) \in D^{n-1} \times D^1. \end{cases}$$

If $\alpha$ has degree $d$, then $g$ twists the handle $d$ times around $D^{n-1} \times 0$. (See Fig. 1 for the case $n = 2$, $d = 1$.)

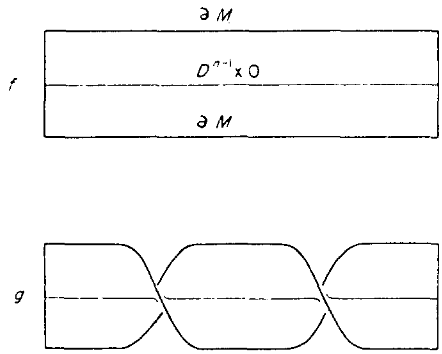


FIG. 1. Images of a handle under $f$ and $g$

Now $\partial M_1$ meets $D^{n-1} \times D^n$ in the union of the images of two antipodal sections, $\phi_+$ and $\phi_-$, of the bundle $D^{n-1} \times \partial D^n \to D^{n-1}$. Likewise, $g(\partial M_1)$ is the union of the images of two antipodal sections $\psi_+$ and $\psi_-$, namely, $\psi_+(x) = (x, \alpha(x))$ and $\psi_-(x) = (x, -\alpha(x))$. The obstruction to deforming $\phi_+$ into $\psi_+(\text{rel } \partial D^{n-1})$ is the homotopy class $\{\alpha\} \in \pi_{n-1}(S^{n-1})$, and so is the obstruction to deforming $\phi_-$ into $\psi_-(\text{rel } \partial D^{n-1})$.

To compute $u_g$, we form a singular disk $C'$ bounded by $g(\partial M_0)$ by adjoining to $C$ the images $Y_+$, $Y_-$ of two homotopies in $D^{n-1} \times D^n$ that take $\phi_+$ and $\phi_-$ into $\psi_+$ and $\psi_-$ respectively. From (3.3) we see that

$$C \# (D^{n-1} \times 0) - C' \# (D^{n-1} \times 0) = (Y_+ \# (D^{n-1} \times 0)) + (Y_- \# (D^{n-1} \times 0)) = 2d,$$

where $d$ is the degree of $\alpha$.

Since $d$ is an arbitrary integer, we can choose $g$ so that the homology class $[u_g]$ vanishes (assuming that $[u_f]$ is uneven). This completes the proof of 2.1.

(4.3). *Proof of* (2.2). We keep the notation of (4.1), except as otherwise indicated. Let $f: M \to R^{2n-k}$ be an embedding, and let $\varepsilon$ be the radius of a tubular neighborhood of $f(M)$. If $v$ is a normal vector field on $f(M_0)$, let $f_v: M \to R^{2n-k}$ be the map defined by

$$f_v(x) = \begin{cases} f(x) + \lambda(x)v(x) \text{ if } x \in M_1 \\ f(x) \text{ if } x \in D_1. \end{cases}$$

First of all we have to define the correspondence $\Phi$ of Theorem (2.2). We claim that *if $f: M \to R^{2n-k}$ is an embedding, there exists a normal vector field $v$ on $f(M_0)$ such that $f_v(M)$ is homologous to zero in $X = R^{2n-k} - f(M_2)$, and any two such normal vector fields are homotopic.*

An argument like that in (4.1) shows that $X$ is $(n-1)$-connected, and $\pi_n(X) \approx H_n(X) \approx H^{n-k-1}(M_2)$. If $v$, $v'$ are any two vector fields normal to $f(M_2)$, the difference class $d(v, v') \in H^{n-k-1}(M_2)$ corresponds to the homology class $[f_v(M)] - [f_{v'}(M)] \in H_n(X)$, under Alexander duality, according to (3.3). (The orientability of $M$ is used here.) Since the homotopy classes of normal vector fields on $f(M_0)$ are in 1-1 correspondence with $H^{n-k-1}(M_0) \approx H^{n-k-1}(M_2) \approx H_n(X)$, there is one and only one normal vector field $v$, up to homotopy, such that $f_v(M)$ is homologous to zero in $X$.

The correspondence associating to $f$ the couple $(f|M_0, v)$ induces a correspondence $\Phi$ which to the isotopy class of the embedding $f: M \to R^{2n-k}$ assigns the regular homotopy class of the immersion $f|M_0$ with the normal vector field $v$.

(a) $\Phi$ *is injective.* Let $f, g: M \to R^{2n-k}$ be two embeddings, and let $v$, $\mu$ be the normal vector fields to $f^0 = f|M_0$ and $g^0 = g|M_0$ associated as before to $f$ and $g$. Suppose that $(f^0, v)$ and $(g^0, \mu)$ are regularly homotopic. By (3.1) we can assume they are isotopic.

Let $h_t: M_0 \to R^{2n-k}$ be an isotopy such that $h_0 = f^0$ and $h_1 = g^0$, and let $\lambda_t$ be a normal vector field on $h_t(M_0)$ with $\lambda_0 = v$ and $\lambda_1 = \mu$

We may thus assume that $f$ and $g$ agree on $M_1$, and that $v = \mu$, because an isotopy of $h_0(M_0)$ can be extended to an isotopy of $R^{2n-k}$; cf. [11], [12]. Since $f_v(M)$ and $g_\mu(M)$ are homologous to zero in $X = R^{2n-k} - f(M_2) = R^{2n-k} - g(M_2)$, we see that $f_v(M) - g_\mu(M) \sim 0$ and hence $f(D_1) - g(D_1) \sim 0$. Thus $f|D_1$ and $g|D_1$ are homotopic (rel $\partial D_1$) in $X$. By (3.2) they are isotopic in $X$ by an isotopy fixed on a neighborhood of $\partial D_2$. Hence $f$ and $g$ are isotopic.

(b) $\Phi$ *is surjective.* Let $f^0: M_0 \to R^{2n-k}$ be an immersion with a normal vector field $v$ As in (4.1), we can assume (by 2.1) that $f^0$ is an embedding. Put $X = R^{2n-k} - f^0(M_2)$.

Since $\pi_n(X) \approx H_n(X)$, the map $x \to f^0(x) + \lambda(x)v(x)$ of $M_1$ in $R^{2n-k}$ can be extended to a map $f_v: M \to X$ such that $f_v(M) \sim 0$ in $X$. Let $g: D_2 \to X$ be defined by

$$g(x) = \begin{cases} f^0(x) \text{ if } x \in D_2 - D_1 \\ f_v(x) \text{ if } x \in D_1. \end{cases}$$

As in (4.1), it follows from (3.2) that we can obtain an embedding $f: M \to R^{2n-k}$ such that $f_v(M) \sim 0$ in $X$.

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